MATH3705 Tutorial 3

1. Consider the DE: $y'' + 2xy' + y = 0$.

(a) Find the general solution of the equation about $x_0 = 0$.

(b) Solve the initial-value problem if $y(0) = 2$ and $y'(0) = -3$.

Solution: (a) 1) Consider singularities: none. So we shall attempt to find a solution in the form of an infinite series.

2) Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \Rightarrow xy' = \sum_{n=1}^{\infty} n a_n x^n,$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_n x^n.$$

3) We imply that

$$a_{n+2} = -\frac{2n+1}{(n+1)(n+2)} a_n,$$

which gives,

$$a_{2n} = (-1)^n \frac{1 \cdots 5 \cdot 9 \cdots (4n-4)}{(2n)!} a_0,$$

$$a_{2n+1} = (-1)^n \frac{3 \cdots 7 \cdot 11 \cdots (4n-1)}{(2n+1)!} a_1.$$

4) We have

$$y = a_0 \left(1 - \frac{1}{2} x^2 + \sum_{n=2}^{\infty} (-1)^n \frac{1 \cdots 5 \cdot 9 \cdots (4n-4)}{(2n)!} x^{2n} \right)$$

$$+ a_1 \left(x + \sum_{n=1}^{\infty} (-1)^n \frac{3 \cdots 7 \cdot 11 \cdots (4n-1)}{(2n+1)!} x^{2n+1} \right)$$

2. Find the general solution for $x \neq 0$.

(a) $x^2y'' - 3xy' - 12y = 0$

Solution: The indicial equation is

$$r^2 - 4r - 12 = 0, \Rightarrow r = -2, 6.$$

The general solution is

$$y(x) = c_1|x|^{-2} + c_2|x|^6.$$

(b) $4x^2y'' - 16xy' + 25y = 0$

Solution: We change the equation to $x^2y'' - 4xy' + \frac{25}{4}y = 0$. The indicial equation is

$$r^2 - 5r + 2.5^2 = 0, \Rightarrow r = 2.5.$$

The general solution is

$$y(x) = c_1|x|^{2.5} + c_2|x|^{2.5}\ln|x|.$$

(c) $x^2y'' + 9xy' + 17y = 0$

Solution: The indicial equation is

$$r^2 + 8r + 17 = 0, \Rightarrow r = -4 \pm i.$$

The general solution is

$$y(x) = c_1|x|^{-4}\cos(\ln|x|) + c_2|x|^{-4}\sin(\ln|x|).$$

3. Find the general solution of

$$2x^2y'' + 7xy' + (3 + 2x)y = 0 \qquad (II.1)$$

for $x > 0$ near $x_0 = 0$.

Solution: Step 1: Determine whether $x_0 = 0$ is an ordinary point or a regular singular point. Write the equation as

$$y'' + (\frac{7}{2x}y' + \frac{3+2x}{2x^2}y = 0, \qquad (II.2)$$

We have

$$xp(x) = 3.5, x^2 q(x) = 1.5 + x.$$

So $x_0 = 0$ is a regular singular point.

Step 2: Find and solve the indicial equation. Note that $p_0 = 3.5, q_0 = 1.5$. The indicial equation is:

$$r^2 + (p_0 - 1)r + q_0 = 0. \Rightarrow r^2 + 2.5r + 1.5 = 0. \Rightarrow r_1 = -1, r_2 = -1.5.$$

Note that $r_1 - r_2 = 0.5$, so we have Case (i).

Step 3: Find the recursive relation about $c_n(r)$. Let

$$y = \sum_{n=0}^{\infty} c_n(r) x^{n+r}, \quad c_0(r) = 1. \Rightarrow$$

$$y' = \sum_{n=0}^{\infty} (n+r) c_n(r) x^{n+r-1}, y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n(r) x^{n+r-2}.$$

Substitute them into (II.1) we have

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n(r) x^{n+r-2} + 7x \sum_{n=0}^{\infty} (n+r) c_n(r) x^{n+r-1} + (3+2x) \sum_{n=0}^{\infty} c_n(r) x^{n+r} = 0,$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) c_n(r) x^{n+r} + \sum_{n=0}^{\infty} 7(n+r) c_n(r) x^{n+r}$$

$$+ \sum_{n=0}^{\infty} 3 c_n(r) x^{n+r} + \sum_{n=0}^{\infty} 2 c_n(r) x^{n+r+1} = 0,$$

$$\sum_{n=0}^{\infty} [2(n+r)(n+r-1) + 7(n+r) + 3] c_n(r) x^{n+r} + \sum_{n=0}^{\infty} 2 c_n(r) x^{n+r+1} = 0,$$

$$\sum_{n=0}^{\infty} [(2n+2r+3)(n+r+1)] c_n(r) x^{n+r} + \sum_{n=1}^{\infty} 2 c_{n-1}(r) x^{n+r} = 0,$$

$$(2r+3)(r+1)] c_0(r) x^r + \sum_{n=1}^{\infty} \{[(2n+2r+3)(n+r+1)] c_n(r) + 2c_{n-1}(r)\} x^{n+r} = 0,$$

$$(2r+3)(r+1)] c_0(r) = 0, \quad [(2n+2r+3)(n+r+1)] c_n(r) + 2c_{n-1} = 0, n \geq 1.$$

Since $c_0(r) = 1 \neq 0$, the above first equation results in our indicial equation $(2r+3)(r+1) = 0$. The second equation gives

$$c_n(r) = \frac{-2}{(2n+2r+3)(n+r+1)} c_{n-1}(r), n \geq 1. \qquad (II.3)$$

Step 4: Find y_1. Take $r = r_1 = -1$, by (II.3),

$$c_n(-1) = \frac{-2}{n(2n+1)} c_{n-1}(-1), n \geq 1. \Rightarrow$$

$$c_n(1) = \frac{(-2)^n}{n!(2n+1)!!}, n \geq 1.$$

Therefore,

$$y_1 = x^{-1} + \sum_{n=1}^{\infty} \frac{(-2)^n}{n!(2n+1)!!} x^{n-1} = x^{r_1} \left(1 + \sum_{n=1}^{\infty} \frac{(-2)^n}{n!(2n+1)!!} x^n \right).$$

Step 5: Find y_2. Take $r = r_2 = -1.5$, by (II.3),

$$c_n(-1.5) = \frac{-2}{n(2n-1)} c_{n-1}(-1.5). \Rightarrow$$

$$c_n(-1.5) = \frac{(-2)^n}{n!(2n-1)!!}.$$

Therefore,

$$y_2 = x^{-1.5} + \sum_{n=1}^{\infty} \frac{(-2)^n}{n!(2n-1)!!} x^{n-1.5}.$$

Step 6: The general solution is $y(x) = c_1 y_1 + c_2 y_2$, where c_1 and c_2 are constants.

4. Find two linearly independent solutions $x^2 y''(x) + x y'(x) + (3x^2 - 4) y(x) = 0$, valid for $x > 0$:

Solution: Note that $\lambda^2 = 3$ and $\nu^2 = 4$, $\Rightarrow \lambda = \sqrt{3}$ and $\nu = 2$. Hence

$$y_1(x) = J_2(\sqrt{3}x), \quad y_2(x) = Y_2(\sqrt{3}x).$$